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Boley's method for two-dimensional thermoelastic problems applied to piezoelastic structures ☆

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Abstract

We study a composite piezoelastic plate in cylindrical bending. The plate is composed of perfectly bonded substrate and piezoelastic layers. We assume a plane state of strain; the plate mid-plane deforms into a cylindrical surface perpendicular to, and the electric field vector lies in, the (x,z)-plane. We utilize Bruno Boley's method for two-dimensional thermoelastic problems: Boley introduced an expansion of the Airy stress function and a step-by-step solution for each term. Furthermore, he discussed relations to strength-of-material theories.

For the piezoelastic problem we apply Boley's method to the charge equations of electrostatics. The electric potential is expanded and each term is calculated using a step-by-step procedure. Use of Boley's method is facilitated by the capabilities of modern symbolic computer codes. We solve the problem for an arbitrary distribution of strain first; then we consider the variation of displacements in the form of a third order power series expansion in the z-direction. Strength-of-material theories of different approximation levels are finally extracted, for which the level of approximation for the mechanical and electric field is not independent.

Keywords: Thermoelastic problems; Boley's method; Application to piezoelastic structures

1. Introduction

Over the last few decades piezoelastic materials have become prominent in the fields of mechatronics, structronic systems and electro-mechanics, see Tani et al. (1998) or Tzou (1998) for reference. Piezoelectric solids are utilized to realize distributed actuators and sensors for vibration control of flexible structures, cf. Rao and Sunar (1994). In the high-end technological concept of "intelligent" or "smart" structures, piezoelectric sensors and actuators serve as integrated parts of the structure and are combined with automatic

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control systems, such that the structure is capable of reacting to external disturbances similar to an intelligent being. Frequently, smart structures are realized by means of thin piezoelastic layers equipped with electrodes mounted at their surfaces. These layers are embedded in or attached to substrate layers, resulting in a laminate structure. Applying an electric potential difference at the electrodes, an electric field emerges within the piezoelectric layer due to the converse piezoelectric effect, generally resulting in deformation or mechanical stress. Conversely, a deformation of the structure produces an electric field within the piezoelectric layers. This latter effect is called the direct piezoelectric effect. The piezoelectric effects result in a coupling between mechanical and electrical fields. It is important for practical problems, e.g. in the field of active control of structures, to include electro-mechanical coupling into the modeling in order to obtain an acceptable level of accuracy.

A crucial point in the modeling of piezoelectric laminates is to choose an appropriate approximation for the thickness distribution of the displacement and the electric potential. In order to incorporate the variation of mechanical fields and electric fields accurately, numerous theories have been developed. In equivalent single layer theories displacements are expanded into power series in the thickness direction, see Reddy (1989) for the non-piezoelastic case. Electrical fields for each layer are also expanded into power series in the thickness direction with terms up to the third order, see e.g. Tiersten (1993), Yang and Batra (1994) and Yang (1999). Different approaches can be found in Fernandes and Pouget (2001), accounting for thickness variations by means of harmonic functions, or in Batra and Vidoli (2002), where Legendre polynomials are used. Also, discrete layerwise theories and hybrid or mixed formulations can be widely found in the literature, for example Tzou and Ye (1996), Lee and Saravanos (1997) and Mitchell and Reddy (1995).

In the present paper we restrict our attention to equivalent single layer theories using expansions of displacements into power series. Given the order of the approximation for the displacement, we seek an appropriate expansion of the electric potential inside each piezoelectric layer. Both the order of the expansion and the basis functions of the expansion should be chosen to result in a consistent electromechanically coupled theory. To find appropriate approximations for the electric potential we use an elegant and valuable method, which was originally developed by Bruno Boley, see Boley (1956) and Boley and Weiner (1960). Boley's method is a general analytical successive-approximation method for the solution of linear partial differential equations. The method is applicable when solutions are desired for bodies with one dimension small compared to the others, as pointed out by Boley himself. Originally this method was applied to two-dimensional thermoelastic problems.

In the present paper we consider the cylindrical bending of moderately thick laminated plates. We apply Boley's method to the charge equations of electrostatics. The electric potential is expanded and each term is calculated by a step-by-step procedure. Solving this problem by Boley's method is straightforward, and the use of Boley's method is facilitated by the capabilities of modern symbolic computer codes. We solve the problem for an arbitrary distribution of strain in a first step. Then we consider the variation of displacements in the form of a power series expansion with respect to the thickness direction. Terms up to the third order are taken into account such that the classical theory and the first order, the second order and the third order shear deformation theories are represented.

A cascade of consistent strength-of-material theories of different approximation levels is finally extracted by taking into account the relative thinness of the piezoelastic layers as a characteristic parameter. We consider the Kirchhoff theory and the Reissner–Mindlin theory as special examples (Kirchhoff, 1850; Reissner, 1944, 1945; Hencky, 1947; Mindlin, 1951). Two principal results are derived. The first of the principal results obtained are theories and formulas for the analysis of this type of structure, which can be used to incorporate the coupling by means of effective stiffness parameters. These theories leave the formal structure of the mechanical theory unchanged. In case of very thin piezoelastic layers a sufficient accuracy is obtained, see Krommer and Irschik (1999) and Krommer (2001). The second of the principal results, which comes into the play for moderately thick piezoelastic layers, are appropriate approximations for the thickness distribution of the electric potential inside a piezoelastic layer. In combination with variational principles consistent strength-of-material theories can be derived, which also include field equations for the electric potential, cf. Krommer and Irschik (2002). To the opinion of the authors these principal results should be considered in the modeling of piezoelastic laminates.

2. Mathematical modeling

For a material with the symmetry properties of an orthorhombic system of class 2 mm, the linearized three-dimensional constitutive relations for the electric displacement vector can be written in technical notations as

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} + \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{1}$$

see Eringen and Maugin (1990). Consider a composite plate with the reference surface in the (x, y)-plane. The deformation is assumed to take place in the (x, z)-plane only, resulting in a plane state of strain. The strain components ε_{yy}, γ_{xy}, γ_{yz} therefore vanish and the electric field vector lies in the (x, z)-plane. We refer to an arbitrary piezoelastic layer located at $z_1 \leqslant z \leqslant z_2$ in the remainder of this paper. The non-vanishing components of the electric displacement vector D_x, D_z in this layer are

$$D_x = e_{xz}\gamma_{xz} + \epsilon_x E_x \quad D_z = e_x\varepsilon_{xx} + e_z\varepsilon_{zz} + \epsilon_z E_z \tag{2}$$

with the piezoelectric coefficients and the electric permittivites

$$e_{xz} = e_{15} \quad e_x = e_{31} \quad e_z = e_{33} \quad \epsilon_x = \epsilon_{11} \quad \epsilon_z = \epsilon_{33} \tag{3}$$

With the aid of the two-dimensional charge equation of electrostatics

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = 0 \tag{4}$$

a second order linear partial differential equation for the electric potential ϕ, which defines the electric field as its negative gradient, is found

$$\epsilon_x \frac{\partial^2 \phi}{\partial x^2} + \epsilon_z \frac{\partial^2 \phi}{\partial z^2} = e_x \frac{\partial \varepsilon_{xx}}{\partial z} + e_{xz} \frac{\partial \gamma_{xz}}{\partial x} + e_z \frac{\partial \varepsilon_{zz}}{\partial z} \tag{5}$$

Note that material parameters have been assumed to be constant within the layer.

3. Method of solution

To find a solution of Eq. (5) we apply a method originally developed by Bruno Boley for thermoelastic problems, see Boley (1956). We write the governing equation for the electric potential as

$$(D_{\phi x} + D_{\phi z})\phi(x, z) = D_{\phi\varepsilon x}\varepsilon_{xx}(x, z) + D_{\phi\varepsilon xz}\gamma_{xz}(x, z) + D_{\phi\varepsilon z}\varepsilon_{zz}(x, z) \tag{6}$$

where the differential operators, which contain derivatives with respect to one coordinate only, are defined by

$$D_{\phi x} = \epsilon_x \frac{\partial^2}{\partial x^2} \quad D_{\phi z} = \epsilon_z \frac{\partial^2}{\partial z^2} \quad D_{\phi \varepsilon x} = e_x \frac{\partial}{\partial z} \quad D_{\phi \varepsilon xz} = e_{xz} \frac{\partial}{\partial x} \quad D_{\phi \varepsilon z} = e_z \frac{\partial}{\partial z} \tag{7}$$

We now seek a solution for the electric potential in the form

$$\phi(x,z) = \sum_{i=0}^{\infty} \phi_i(x,z) \tag{8}$$

Inserting Eq. (8) into Eq. (6) we find

$$\begin{aligned} &[(D_{\phi x} + D_{\phi z})\phi_0] + [D_{\phi z}\phi_1 - D_{\phi \varepsilon x}\varepsilon_{xx}] + [D_{\phi z}\phi_2 - D_{\phi \varepsilon xz}\gamma_{xz}] + [D_{\phi z}\phi_3 - D_{\phi \varepsilon z}\varepsilon_{zz}] \\ &\quad + [D_{\phi z}\phi_4 + D_{\phi x}(\phi_1 + \phi_2 + \phi_3)] + \sum_{i=5}^{\infty}[D_{\phi z}\phi_i + D_{\phi x}\phi_{i-1}] = 0 \end{aligned} \tag{9}$$

where the functions $\phi_i(x,z)$ are assumed to be governed by

$$\begin{aligned} &(D_{\phi x} + D_{\phi z})\phi_0 = 0 \quad D_{\phi z}\phi_1 = D_{\phi \varepsilon x}\varepsilon_{xx} \quad D_{\phi z}\phi_2 = D_{\phi \varepsilon xz}\gamma_{xz} \quad D_{\phi z}\phi_3 = D_{\phi \varepsilon z}\varepsilon_{zz} \\ &\quad D_{\phi z}\phi_4 = -D_{\phi x}(\phi_1 + \phi_2 + \phi_3) \quad D_{\phi z}\phi_i = -D_{\phi x}\phi_{i-1} \quad i = 5, 6, 7, \ldots \end{aligned} \tag{10}$$

The definition of the portions ϕ_i of the electric potential ϕ is not unique. The motivation for the above manner of choosing the functions ϕ_i will become obvious in the next section. It remains to formulate the boundary conditions. Extension of the plate in x-direction is L and the total thickness is h. The piezoelastic layer, which is perfectly bonded to the laminate, has a thickness $c = z_2 - z_1$. Its upper face and its lower face are electroded. At the upper face a constant electric potential is prescribed and the lower face is grounded. The boundary conditions at $z = z_1$, z_2 are

$$z = z_1 : \phi = \phi^U \quad z = z_2 : \phi = 0 \tag{11}$$

The faces at $x = 0, L$ are not electroded, thus electric displacement free boundary conditions have to be satisfied

$$x = 0, L : D_x = 0 \Longleftrightarrow \epsilon_x \frac{\partial \phi}{\partial x} = e_{xz}\gamma_{xz} \tag{12}$$

The electric boundary conditions at $z = z_1$, z_2 are taken as

$$z = z_1 : \phi_0 = \phi^U \quad z = z_2 : \phi_0 = 0 \quad z = z_1, z_2 : \phi_i = 0, \quad i = 1, 2, 3, \ldots \tag{13}$$

From Eq. (10) it follows that the functions ϕ_i, $i = 1, 2, 3, \ldots$, cannot be adjusted to the boundary conditions at $x = 0, L$. These latter boundary conditions are therefore accounted for by means of ϕ_0 in the form

$$\epsilon_x \frac{\partial \phi_0}{\partial x} = e_{xz}\gamma_{xz} - \sum_{i=1}^{\infty} \epsilon_x \frac{\partial \phi_i}{\partial x} \tag{14}$$

4. Solution for the electric potential

In order to simplify the following calculations, we assume the origin of the thickness coordinate to be located at the location of the upper electrode of the layer. Hence $z_1 = 0$ and $z_2 = c$. We split the solution for ϕ_0 into two parts. The first part ϕ_{01} accounts for the non-homogenous boundary conditions at $z = 0, c$, whereas the second part ϕ_{02} accounts for the non-homogenous boundary conditions at $x = 0, L$. ϕ_{02} cannot be calculated as long as all the other terms of the expansion have been calculated. We therefore consider ϕ_{01}

in a first step of the solution procedure. The solution for ϕ_{01} corresponds to the simple model of a capacitance. We have

$$\phi_{01}(z) = \phi^U\left(1 - \left(\frac{z}{c}\right)\right) \tag{15}$$

In many practical applications of piezoelastic structures, the approximation of Eq. (15) for the electric potential is used, neglecting the influence of the direct piezoelastic effect. Due to Tiersten (1969) this approximation is denoted as small piezoelectric coupling.

We proceed by calculating the solution for the first term, the second term and the third term of the series expansion ϕ_1, ϕ_2 and ϕ_3. These terms are denoted as the elementary influences of the axial normal strain ε_{xx}, of transverse shear strain γ_{xz} and of transverse normal strain ε_{zz}. The governing ordinary differential equations are $D_{\phi z}\phi_1 = D_{\phi\varepsilon x}\varepsilon_{xx}$, $D_{\phi z}\phi_2 = D_{\phi\varepsilon xz}\gamma_{xz}$ and $D_{\phi z}\phi_3 = D_{\phi\varepsilon z}\varepsilon_{zz}$, and the Dirichlet boundary conditions at $z = 0, c$ are homogenous. The solutions are

$$\frac{\epsilon_z}{e_x}\phi_1 = \int_0^c \varepsilon_{xx}\,\mathrm{d}\bar{z} - \left(\frac{z}{c}\right)\int_0^c \varepsilon_{xx}\,\mathrm{d}\bar{z} \tag{16a}$$

$$\frac{\epsilon_z}{e_x}\phi_2 = \frac{e_{xz}}{e_x}\frac{\partial}{\partial x}\left[\left(\frac{z}{c}\right)\int_0^z \gamma_{xz}c\,\mathrm{d}\bar{z} - \int_0^z \gamma_{xz}\bar{z}\,\mathrm{d}\bar{z} - \left(\frac{z}{c}\right)\left\{\int_0^c \gamma_{xz}c\,\mathrm{d}\bar{z} - \int_0^c \gamma_{xz}\bar{z}\,\mathrm{d}\bar{z}\right\}\right] \tag{16b}$$

$$\frac{\epsilon_z}{e_x}\phi_3 = \frac{e_z}{e_x}\left[\int_0^z \varepsilon_{zz}\,\mathrm{d}\bar{z} - \left(\frac{z}{c}\right)\int_0^c \varepsilon_{zz}\,\mathrm{d}\bar{z}\right] \tag{16c}$$

Summing up the three terms of Eq. (16), the electric potential distribution we obtain is a solution of the charge equation of electrostatics, in which the axial component E_x of the electric field vector is neglected. Due to this fact we have denoted the distributions of Eq. (16) as elementary influences of strain components. In the next section the applicability of these elementary solutions in connection with strength-of-material theories will be discussed.

To incorporate the influence of E_x we calculate the fourth term of the series expansion, which is governed by $D_{\phi z}\phi_4 = -D_{\phi x}(\phi_1 + \phi_2 + \phi_3)$. It is useful to split the solution for ϕ_4 into three parts, where each part accounts for one of the elementary solutions. In this sense ϕ_4 corrects the elementary solution with respect to the influence of E_x.

$$\frac{\epsilon_z}{e_x}\phi_{41} = -\frac{\epsilon_x}{\epsilon_z}\frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\left(\frac{z}{c}\right)^2\int_0^z \varepsilon_{xx}c^2\,\mathrm{d}\bar{z} - \left(\frac{z}{c}\right)\int_0^z \varepsilon_{xx}\bar{z}c\,\mathrm{d}\bar{z} + \frac{1}{2}\int_0^z \varepsilon_{xx}\bar{z}^2\,\mathrm{d}\bar{z} - \left(\frac{z}{c}\right)\left\{\frac{1}{3}\left(1 + \frac{1}{2}\left(\frac{z}{c}\right)^2\right)\int_0^c \varepsilon_{xx}c^2\,\mathrm{d}\bar{z} - \int_0^c \varepsilon_{xx}\bar{z}c\,\mathrm{d}\bar{z} + \frac{1}{2}\int_0^c \varepsilon_{xx}\bar{z}^2\,\mathrm{d}\bar{z}\right\}\right] \tag{17a}$$

$$\begin{aligned}\frac{\epsilon_z}{e_x}\phi_{42} = -\frac{\epsilon_x}{\epsilon_z}\frac{e_{xz}}{e_x}\frac{\partial^3}{\partial x^3}\Bigg[&\frac{1}{6}\left(\frac{z}{c}\right)^3\int_0^z \gamma_{xz}c^3\,\mathrm{d}\bar{z} - \frac{1}{2}\left(\frac{z}{c}\right)^2\int_0^z \gamma_{xz}\bar{z}c^2\,\mathrm{d}\bar{z} + \frac{1}{2}\left(\frac{z}{c}\right)\int_0^z \gamma_{xz}\bar{z}^2c\,\mathrm{d}\bar{z} \\ &- \frac{1}{6}\int_0^z \gamma_{xz}\bar{z}^3\,\mathrm{d}\bar{z} - \frac{1}{6}\left(\frac{z}{c}\right)\left\{\left(\frac{z}{c}\right)^2\int_0^c \gamma_{xz}c^3\,\mathrm{d}\bar{z} - \left(2 + \left(\frac{z}{c}\right)^2\right)\int_0^c \gamma_{xz}\bar{z}c^2\,\mathrm{d}\bar{z} \\ &+ \int_0^c \gamma_{xz}\bar{z}^2(3c - \bar{z})\,\mathrm{d}\bar{z}\right\}\Bigg]\end{aligned} \tag{17b}$$

$$\begin{aligned}\frac{\epsilon_z}{e_x}\phi_{43} = -\frac{\epsilon_x}{\epsilon_z}\frac{e_z}{e_x}\frac{\partial^2}{\partial x^2}\Bigg[&\frac{1}{2}\left(\frac{z}{c}\right)^2\int_0^z \varepsilon_{zz}c^2\,\mathrm{d}\bar{z} - \left(\frac{z}{c}\right)\int_0^z \varepsilon_{zz}\bar{z}c\,\mathrm{d}\bar{z} + \frac{1}{2}\int_0^z \varepsilon_{zz}\bar{z}^2\,\mathrm{d}\bar{z} \\ &- \left(\frac{z}{c}\right)\left\{\frac{1}{3}\left(1 + \frac{1}{2}\left(\frac{z}{c}\right)^2\right)\int_0^c \varepsilon_{zz}c^2\,\mathrm{d}\bar{z} - \int_0^c \varepsilon_{zz}\bar{z}c\,\mathrm{d}\bar{z} + \frac{1}{2}\int_0^c \varepsilon_{zz}\bar{z}^2\,\mathrm{d}\bar{z}\right\}\Bigg]\end{aligned} \tag{17c}$$

Obviously derivatives with respect to the axial coordinate of two orders higher than in the elementary solution are present in the solutions of Eq. (17). For that reason the three parts of ϕ_4 are denoted as second order corrections of the influences of the different components of the strain tensor.

The terms of the series expansion for the electric potential were calculated by means of modern symbolic computer codes. Calculation of higher order corrections is straightforward; we can expect subsequently higher order corrections, which contain higher order derivatives with respect to the axial coordinate for the influence of the components of the strain tensor. The difficult task in finding the exact solution of the problem is to find a solution for ϕ_{02}, which accounts for the non-homogenous boundary conditions at $x = 0, L$. Thus the solution of a homogenous linear partial differential equation with non-homogenous boundary conditions has to be calculated. However, for a relatively thin structure, the influence of the latter non-homogenous boundary conditions can be intuitively expected to be restricted to the vicinity of the boundary by means of an electric analog to Saint-Venant's principle. This was also noted by Boley (1956) for the thermal problem. In order to justify this assumption the next section is devoted to the application of the approximations for the electric potential to strength-of-material theories.

5. Relations to strength-of-material theories

Relations to strength-of-material theories are established in this section. We consider a general formulation for equivalent single layer theories, unifying the classical theory, the first order, the second order and the third order shear deformation theories, (see Reddy (1989) for a classification.) The displacement is

$$u(x,z) = u_0 + \alpha z \frac{\partial w_0}{\partial x} + \beta z \psi_u + \lambda z^2 \varphi_u + \gamma z^3 \theta_u \quad w(x,z) = w_0 + \lambda z \psi_w + \gamma z^2 \varphi_w \tag{18}$$

Strain components can be calculated from Eq. (18) in the generalized form

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \left(\alpha \frac{\partial^2 w_0}{\partial x^2} + \beta \frac{\partial \psi_u}{\partial x} \right) z + \lambda \frac{\partial \varphi_u}{\partial x} z^2 + \gamma \frac{\partial \theta_u}{\partial x} z^3 \quad \varepsilon_{zz} = \lambda \psi_w + 2\gamma z \varphi_w$$
$$\gamma_{xz} = (1+\alpha) \frac{\partial w_0}{\partial x} + \beta \psi_u + \lambda \left(2\varphi_u + \frac{\partial \psi_w}{\partial x} \right) z + \gamma \left(3\theta_u + \frac{\partial \varphi_w}{\partial x} \right) z^2 \tag{19}$$

Tracers α, β, λ, γ are introduced in Eqs. (18) and (19) to distinguish between different theories. The axial strain ε_{xx} is of order three in powers of z, the transverse shear strain γ_{xz} is of order two in powers of z, and ε_{zz} is of order one in powers of z. In the following we consider ϕ_1, ϕ_2, ϕ_3 and ϕ_4, because ϕ_{01} does not depend on the distribution of strain. The electric potentials ϕ_1 and ϕ_{41} which account for ε_{xx} are presented in Table 1, where the non-dimensional coordinates $\eta = zc^{-1}$ and $\xi = xL^{-1}$ are used. $\Delta = cL^{-1}$ denotes the thickness-to-length ratio of the piezoelastic layer. Each line of Table 1 corresponds to one single power of z present in the distribution of strain. The electric potentials ϕ_2 and ϕ_{42} which account for γ_{xz} are presented in Table 2, again separately for each individual power of z. ε_{zz} is taken into account by ϕ_3 and ϕ_{43}. These electric potentials are presented in Table 3.

Subsequently, we specialize the results of Tables 1–3 to two of the most commonly used strength-of-material theories. In the classical theory of plates, which is due to Kirchhoff (1850), we have $\alpha = -1$, $\beta = \lambda = \gamma = 0$, such that the electric potential becomes

$$\frac{\epsilon_z}{e_x} \phi^K = \left\{ 1 + \frac{\epsilon_x}{\epsilon_z} \frac{\Delta^2}{12} (1 + \eta - \eta^2) \frac{\partial^2}{\partial \xi^2} \right\} \frac{\partial^2 w_0}{\partial \xi^2} \frac{\Delta^2}{2} \eta (1 - \eta) \tag{20}$$

The term with Δ^2 enters via the second order correction calculated in Section 4. For a thin layer $\Delta^2 \ll 1$ can be assumed. Hence we obtain

Table 1
Distribution of electric potentials $\epsilon_z e_x^{-1}(\phi_1 + \phi_{41})$

z^0	0
z^1	$-\left\{1+\frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{12}(1+\eta-\eta^2)\frac{\partial^2}{\partial\xi^2}\right\}\left(\alpha\frac{\partial^2 w_0}{\partial\xi^2}\frac{\Delta}{c}+\beta\frac{\partial\psi_u}{\partial\xi}\right)\frac{c}{2}\Delta\eta(1-\eta)$
z^2	$-\left\{1+\frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{60}(7-3\eta^2)\frac{\partial^2}{\partial\xi^2}\right\}\left(\lambda\frac{\partial\varphi_u}{\partial\xi}\right)\frac{c^2}{3}\Delta\eta(1-\eta^2)$
z^3	$-\left\{\frac{\epsilon_x}{\epsilon_z}\frac{4\Delta^2}{30}(1+\eta)\frac{\partial^2}{\partial\xi^2}+(1+\eta+\eta^2)\left(1-\frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{30}\eta^2\frac{\partial^2}{\partial\xi^2}\right)\right\}\left(\gamma\frac{\partial\theta_u}{\partial\xi}\right)\frac{c^3}{4}\Delta\eta(1-\eta)$

Table 2
Distribution of electric potentials $\epsilon_z e_x^{-1}(\phi_2 + \phi_{42})$

z^0	$-\left\{1+\frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{12}(1+\eta-\eta^2)\frac{\partial^2}{\partial\xi^2}\right\}\frac{e_{xz}}{e_x}\left((1+\alpha)\frac{\partial^2 w_0}{\partial\xi^2}\frac{\Delta}{c}+\beta\frac{\partial\psi_u}{\partial\xi}\right)\frac{c}{2}\Delta\eta(1-\eta)$
z^1	$-\left\{1+\frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{60}(7-3\eta^2)\frac{\partial^2}{\partial\xi^2}\right\}\frac{e_{xz}}{e_x}\lambda\left(2\frac{\partial\varphi_u}{\partial\xi}+\frac{\partial^2\psi_w}{\partial\xi^2}\frac{\Delta}{c}\right)\frac{c^2}{6}\Delta\eta(1-\eta^2)$
z^2	$-\left\{\frac{\epsilon_x}{\epsilon_z}\frac{4\Delta^2}{30}(1+\eta)\frac{\partial^2}{\partial\xi^2}+(1+\eta+\eta^2)\left(1-\frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{30}\eta^2\frac{\partial^2}{\partial\xi^2}\right)\right\}\frac{e_{xz}}{e_x}\gamma\left(3\frac{\partial\theta_u}{\partial\xi}+\frac{\partial^2\varphi_w}{\partial\xi^2}\frac{\Delta}{c}\right)\frac{c^3}{12}\Delta\eta(1-\eta)$

Table 3
Distribution of electric potentials $\epsilon_z e_x^{-1}(\phi_3 + \phi_{43})$

z^0	0
z^1	$-\left\{1+\frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{12}(1+\eta-\eta^2)\frac{\partial^2}{\partial\xi^2}\right\}\frac{e_z}{e_x}2\gamma\varphi_w\frac{c^2}{2}\eta(1-\eta)$

$$\frac{\epsilon_z}{e_x}\phi^K = \frac{\partial^2 w_0}{\partial\xi^2}\frac{\Delta^2}{2}\eta(1-\eta) \tag{21}$$

as an approximation for the electric potential. Neglecting the term with $\Delta^2 \ll 1$, the solution corresponds to the solution of a simplified version of the charge equations of electrostatics, which is obtained by neglecting the influence of E_x

$$E_x \approx 0 : \frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = 0 \Longleftrightarrow \epsilon_z\frac{\partial^2\phi}{\partial z^2} = -e_x\frac{\partial^2 w_0}{\partial x^2} \tag{22}$$

The order of the approximation in Eq. (21) is equal to the order of the approximation of the displacement plus one. The basis function is a polynomial of second order with the coefficients adjusted to give a trivial value at $\eta = 0, 1$. Not calculating higher terms in the expansion for the electric potential in Section 4 is justified, because these terms would enter in Eq. (20) with higher orders in Δ, and therefore can be neglected for the thin layer. It remains to discuss the influence of the non-homogenous boundary conditions at

$x = 0, L$. Obviously the electric boundary conditions at $x = 0, L$ cannot be satisfied by the approximation of Eq. (21). Nevertheless, results calculated by using the approximation of Eq. (21) for a Bernoulli–Euler beam are very good, except near $x = 0, L$, as shown in Krommer (2001) by a comparison to electromechanically coupled two-dimensional Finite Element-computations. Moreover, the mathematical form of the Bernoulli–Euler beam theory is conserved by inserting the approximation of Eq. (21) into definitions of stress resultants. The electromechanical coupling in this approximation comes into the play by means of effective stiffness parameters. The result of Eq. (21) is the first of the principal results that we have mentioned in Section 1. For the classical theory, typically used for thin structures, we content ourselves with this latter result, because of its sufficient accurateness.

In the first order shear deformation theory, developed by Reissner (1944, 1945), Hencky (1947) and Mindlin (1951), we have $\beta = 1$, $\alpha = \lambda = \gamma = 0$, such that the electric potential becomes

$$\frac{\epsilon_z}{e_x}\phi^{\mathrm{RHM}} = -\left[\left\{1 + \frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{12}(1+\eta-\eta^2)\frac{\partial^2}{\partial\xi^2}\right\}\frac{\partial\psi_u}{\partial\xi} + \left\{1 + \frac{\epsilon_x}{\epsilon_z}\frac{\Delta^2}{6}(1+\eta-\eta^2)\frac{\partial^2}{\partial\xi^2}\right\}\frac{e_{xz}}{e_x}\left(\frac{\partial^2 w_0}{\partial\xi^2}\frac{\Delta}{c} + \frac{\partial\psi_u}{\partial\xi}\right)\right] \times \frac{c}{2}\Delta\eta(1-\eta) \tag{23}$$

As for the classical theory it is appropriate to neglect the term with $\Delta^2 \ll 1$ in Eq. (23), because piezoelastic layers are usually thin even in the first order shear deformation theory. Also higher order terms do not have to be accounted for. We find the following approximation for the electric potential distribution by neglecting $\Delta^2 \ll 1$ in Eq. (23):

$$\frac{\epsilon_z}{e_x}\phi^{\mathrm{RHM}} = -\left[\frac{\partial\psi_u}{\partial\xi} + \frac{e_{xz}}{e_x}\left(\frac{\partial^2 w_0}{\partial\xi^2}\frac{\Delta}{c} + \frac{\partial\psi_u}{\partial\xi}\right)\right]\frac{c}{2}\Delta\eta(1-\eta) \tag{24}$$

The solution of Eq. (24) corresponds to the solution of the charge equation of electrostatics, in which E_x is neglected

$$E_x \approx 0 : \frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = 0 \Longleftrightarrow \epsilon_z\frac{\partial^2\phi}{\partial z^2} = e_x\frac{\partial\psi_u}{\partial x} + e_{xz}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial\psi_u}{\partial x}\right) \tag{25}$$

In order to check the applicability of the approximation of Eq. (24), the approximation can be inserted into the definition of stress resultants for the first order shear deformation theory. This latter procedure changes the mathematical form of the first order shear deformation theory. Thus the approximation appears not to be appropriate in this form. Neglecting additionally the influence of transverse shear strain in Eq. (24), resulting in a solution corresponding to $D_z = \text{const.}$, conserves the mathematical form of the first order shear deformation theory, (see Krommer and Irschik (1999) for a detailed analysis). As for the classical theory, effective stiffness parameters characterize the electromechanical coupling. However, such a theory has a number of disadvantages. Most important electric boundary conditions at $x = 0, L$ cannot be satisfied; thus electric charge is not conserved. For the first order shear deformation theory it is no longer acceptable not to account for the boundary conditions at $x = 0, L$. Nevertheless, in context of strength-of-material theories, calculating ϕ_{02} is not adequate. Eq. (24) indicates that an appropriate approximation of the electric potential in the thickness direction should be a polynomial of second order in z, adjusted to give a trivial value at $\eta = 0, 1$. We conclude that in a piezoelastic laminate theory consistent with the first order shear deformation theory, the electric potential should be taken as

$$\phi(x,z) = \phi^U\left(1 - \left(\frac{z}{c}\right)\right) - \chi(x)\left(\frac{z}{c}\right)\left(1 - \left(\frac{z}{c}\right)\right) \tag{26}$$

where the influence of the non-homogenous boundary conditions at $z = 0, c$ has been included. Field equations and boundary conditions for $\chi(x)$ can be obtained from variational principles, see Krommer and

Irschik (2002). For the first order shear deformation theory, the approximation of Eq. (26) represents the second principal result mentioned in Section 1.

5.1. Generalized approximation for the electric potential

In Section 1, two principle results were mentioned. The first of these results are theories and formulas for the analysis of this type of composite structures, which can be used to incorporate the coupling by means of effective stiffness parameters. For the classical theory and the first order shear deformation these theories and formulas were discussed. In the context of the generalized equivalent single layer theory, given by Eq. (18), approximations of this simple type can be easily derived from Tables 1–3 by neglecting terms containing $\Delta^2 \ll 1$. By doing so we obtain

$$\begin{aligned}\phi^{\text{appr}}(x,z) = &-\left((1+\alpha)\frac{e_{xz}}{e_x}\frac{\partial^2 w_0}{\partial x^2} + \alpha\frac{\partial^2 w_0}{\partial x^2} + \beta\frac{\partial \psi_u}{\partial x}\left(1+\frac{e_{xz}}{e_x}\right) + \gamma 2\frac{e_z}{e_x}\varphi_w\right)\frac{c^2}{2}\left(\frac{z}{c}\right)\left(1-\left(\frac{z}{c}\right)\right)\\ &-\lambda\left(\frac{\partial \varphi_u}{\partial x}\left(1+\frac{e_{xz}}{e_x}\right) + \frac{1}{2}\frac{e_{xz}}{e_x}\frac{\partial^2 \psi_w}{\partial x^2}\right)\frac{c^3}{3}\left(\frac{z}{c}\right)\left(1-\left(\frac{z}{c}\right)^2\right)\\ &-\gamma\left(\frac{\partial \theta_u}{\partial x}\left(1+\frac{e_{xz}}{e_x}\right) + \frac{1}{3}\frac{e_{xz}}{e_x}\frac{\partial^2 \varphi_w}{\partial x^2}\right)\frac{c^4}{4}\left(\frac{z}{c}\right)\left(1-\left(\frac{z}{c}\right)^3\right)\end{aligned} \tag{27}$$

The second principal result was concerned with finding an appropriate expansion of the electric potential inside each piezoelectric layer. For this latter expansion, both the order of the expansion and the basis functions of the expansion should be chosen to result in a consistent electromechanically-coupled theory. Eq. (26) gives the result for the first order shear deformation theory. For the generalized equivalent single layer theory, the expansion follows directly from Eq. (27) by replacing all functions, which depend on the axial coordinate, by arbitrary functions. The result reads as follows

$$\begin{aligned}\phi(x,z) = &\phi^U\left(1-\left(\frac{z}{c}\right)\right) - ((1+\alpha)+\alpha+\beta+\gamma)\chi_2(x)\left(\frac{z}{c}\right)\left(1-\left(\frac{z}{c}\right)\right) - \lambda\chi_3(x)\left(\frac{z}{c}\right)\left(1-\left(\frac{z}{c}\right)^2\right)\\ &-\gamma\chi_4(x)\left(\frac{z}{c}\right)\left(1-\left(\frac{z}{c}\right)^3\right)\end{aligned} \tag{28}$$

Eq. (28) represents the approximation sought in the present paper. Each term of order n and with the basis function z^n in the approximation for the normal strains is reflected by a term of order $n+1$ with basis function $\eta(1-\eta^n)$ in the approximation for the electric potential, except terms of order zero, which have no influence on the electric potential distribution. Each term of order n and with the basis function z^n in the approximation for the transverse shear strain is reflected by a term of order $n+2$ with basis function $\eta(1-\eta^{n+1})$ in the approximation for the electric potential. Although we restricted our attention to equivalent single layer theories, the results of this section are also valid for discrete layer-wise theories, which apply the generalized displacement field of Eq. (18) to each individual layer. It is furthermore interesting that in a layer-wise electromechanically-coupled theory the shear stress continuity condition at the layer interface is not influenced by the electric field, because the axial component of the electric field vector vanishes at the location of the electrode.

6. Conclusion

This paper sought to find appropriate approximations for the electric potential in cases, for which the mechanical approximation for the displacement is given in advance. Particular focus evaluated equivalent single layer theories, which expand the thickness distribution of the displacement into power series with terms up to the third order. For the sake of finding these appropriate approximations we utilized a method that was originally developed by Bruno Boley for two-dimensional thermoelastic problems, which we applied to the charge equation of electrostatics. We obtained two principal results. The first result was formulas and theories, which conserve the mathematical form of the uncoupled theory. These theories account for the influence of electromechanical coupling by means of effective stiffnesses, with the disadvantage that electric boundary conditions at $x = 0, L$ are not satisfied in general. The second result was approximations for the electric potential, which when utilized in variational principles, result in consistent electromechanically coupled laminate theories. These approximations can be utilized directly in discrete layer-wise theories. Moreover, Boley's method may be applied to higher-order mechanical theories and to theories utilizing other basis functions. To the opinion of the authors the two principal results should be considered in the modeling of piezoelastic laminates.

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